The Quantifier Exchange rules

In SL we had shortcut rules for allowing us to deal with the negations of $\rightarrow$, v, and & sentences. Similarly, we will have a shortcut rule allowing us to more easily deal with negated quantifier sentences.

The two equivalences that the rules are derived from are:

$\sim\forall xPx$ is equivalent to $\exists x\sim Px$
$\sim\exists xPx$ is equivalent to $\forall x\sim Px$

In each case the negation sign gets 'pushed in' to the other side of the quantifier and the quantifier is changed. Remembering that $\forall x$ is a generalized conjunction and $\exists x$ is disjunction, it can be seen that these rules are generalized rules instances of DeMorgan's Laws. Also, the formula following the quantifier in each case is irrelevant. For example, $\sim\forall x\exists yRxy$ is equivalent to $\exists x\sim\exists yRxy$ and $\sim\exists z(Pz \ \& \ \forall y(Py \rightarrow y=z))$ is equivalent to $\forall z\sim(Pz \ \& \ \forall y(Py \rightarrow y=z))$.

Note that this rule can be used to show the equivalence of the following:

$\sim\forall x\sim Px$ is equivalent to $\exists xPx$
$\sim\exists x\sim Px$ is equivalent to $\forall xPx$

Take the first case. By QE $\sim\forall x\sim Px$ transforms into $\exists x\sim\sim Px$. It is obvious that this is equivalent to $\exists xPx$ (plug in a letter, use a double negation rule and take out the letter). As with other rules, I will allow you to simply skip writing '~~' and go straight to $\exists xPx$ if you wish.

Here is an theorem that is fairly tricky to prove with just the primitive rules: $\exists x(Px\rightarrow\forall yPy)$. However, it is not that difficult to prove using QE.

1	(1) $\sim\exists x(Px\rightarrow\forall yPy)$	A
1	(2) $\forall x\sim(Px\rightarrow\forall yPy)$	1 QE
1	(3) $\sim(Pa\rightarrow\forall yPy)$	2 $\forall$E
1	(4) $Pa \ \& \sim\forall yPy$	3 Neg $\rightarrow$
1	(5) Pa	5 &E
1	(6) $\sim\forall yPy$	4 &E
1	(7) $\forall yPy$	5 $\forall$I
	(8) $\exists x(Px\rightarrow\forall yPy)$	6,7 RAA (1)

Here is another example: $\exists xPx \vee \exists xQx \vdash \exists x(Qx \vee Px)$

Since there is no obvious way to start, I will attempt to prove this by RAA. Without QE I would assume each disjunct of 1 and show

1	(1) $\exists xPx \vee \exists xQx$	A
2	(2) $\sim\exists x(Qx \vee Px)$	A
2	(3) $\forall x\sim(Qx \vee Px)$	2 QE

that each led to a contradiction. This would involve two uses of $\exists$E and is a little tricky. Using QE makes it a little quicker and possibly more straightforward.

2	(4) $\sim(Qa \vee Pa)$	3 $\forall$E
2	(5) $\sim Qa \ \& \sim Pa$	4 DeM
2	(6) $\sim Qa$	5 &E
2	(7) $\sim Pa$	5 &E
2	(8) $\forall x{\sim}Qx$	6 $\forall$I
2	(9) $\sim\exists xQx$	8 QE
1,2	(10) $\exists xPx$	1,9 vE
2	(11) $\forall x{\sim}Px$	7 $\forall$I
2	(12) $\sim\exists xPx$	11 QE
1	(13) $\exists x(Qx \vee Px)$	10,12 RAA(2)

The QE rule allows us to prove that different ways of saying "Nothing" are equivalent. For example, $\forall x(Px \rightarrow Qx)$ and $\sim\exists x(Px \ \& \ Qx)$ are both ways of saying "No Ps are Qs." It is now easy to see why. Starting with the second sentence, $\sim\exists x(Px \ \& \ Qx)$ is equivalent to $\forall x{\sim}(Px \ \& \ Qx)$ by QE which is equivalent to $\forall x(Px \rightarrow Qx)$ by the NegCon rule. This principle generalizes to complicated examples:

At most two Ps can be written as:

$\sim\exists x\exists y\exists z((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$
or as $\forall x\forall y\forall z((Px \ \& \ Py \ \& \ Pz) \rightarrow (x{=}y \vee y{=}z \vee x{=}z))$

Start with the top.
$\sim\exists x\exists y\exists z((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$
is $\forall x{\sim}\exists y\exists z((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$ by QE
is $\forall x\forall y{\sim}\exists z((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$ by QE (changing $\sim\exists y$ into $\forall y{\sim}$)
is $\forall x\forall y\forall z{\sim}((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$ by QE (changing $\sim\exists z$ into $\forall z{\sim}$)

To be correct this requires us to be able to use QE on formulas that are inside a larger formula. Like all equivalence rules, this is a valid step although it would require further proof. Here I will simply assert that this is true.

$\forall x\forall y\forall z{\sim}((Px \ \& \ Py \ \& \ Pz) \ \& \ (x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$
is $\forall x\forall y\forall z{\sim}(Px \ \& \ Py \ \& \ Pz) \vee {\sim}(x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$ by DeMorgan's
is $\forall x\forall y\forall z((Px \ \& \ Py \ \& \ Pz) \rightarrow {\sim}(x{\neq}y \ \& \ y{\neq}z \ \& \ x{\neq}z))$ by the v$\rightarrow$ rule
is $\forall x\forall y\forall z((Px \ \& \ Py \ \& \ Pz) \rightarrow ({\sim}x{\neq}y \vee {\sim}y{\neq}z \vee {\sim}x{\neq}z))$ by DeMorgan's
is $\forall x\forall y\forall z((Px \ \& \ Py \ \& \ Pz) \rightarrow (x{=}y \vee y{=}z \vee x{=}z))$ by simply removing the double negations in the consequent.

Since each of these steps was an equivalence, the process is completely reversible and so the two sentences are equivalent to each other.